

SLE curves and natural parametrization

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Abstract

Developing the theory of two-sided radial and chordal *SLE*, we prove that the natural parametrization on *SLE* _{κ} curves is well defined for all $\kappa < 8$. Our proof uses a two-interior-point local martingale.

Key words and Phrases: *SLE*, natural parametrization, Doob-Meyer decomposition, local martingale

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1 Introduction

1.1 Background and Motivation

Suppose that $x_j, j = 1, 2, \dots$ are independent and identically distributed random vectors in \mathbb{Z}^2 with probabilities

$$\mathbb{P}\{x_j = e\} = 1/4, \quad |e| = 1.$$

It is well known that the scaled simple random walk in \mathbb{R}^2 ,

$$B_t^{(n)} = n^{-1/2} \sum_{j=1}^{\lfloor nt \rfloor} x_j, \quad 0 \leq t \leq 1,$$

converges to a standard two-dimensional Brownian motion $B_t, 0 \leq t \leq 1$ as $n \rightarrow \infty$. In the scaled walk, each step is traversed in the same amount of time. When passing to the scaling limit, this parameter t becomes the natural parametrization of Brownian motion. We can write the scaling factor as $n^{-1/d}$, where $d = 2$ is the fractal dimension of the Brownian paths. The natural parametrization is a d -dimensional measure.

One variant of simple random walk is the loop erased random walk first appeared in [Law80]. Its definition is as follows. Consider any finite or recurrent connected graph G ,

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one vertex a and a set of vertices V . Loop-erased random walk (LERW) from a to V is a random simple curve joining a to V obtained by erasing the loops in chronological order from a simple random walk started at a and stopped upon hitting V . One can ask whether or not there is a corresponding result for LERW where the scaling factor is $n^{-1/d}$ and d is the fractal dimension of the paths. Schramm [Sch00] introduced a process, now called the Schramm-Loewner evolution (SLE_κ), as a candidate for the scaling limit and gave a strong argument why SLE_2 should be the scaling limit of LERW. In order to use the Loewner equation, he used a capacity parametrization which is not the parametrization one would obtain by taking the limit above. In [LSW04], it was proved that the scaling limit of planar LERW *in the capacity parametrization* is SLE_2 . However, it is still open whether or not, one can take a limit as above. There are a number of other models that are known to converge to SLE_κ in the scaling limit using the capacity parametrization: critical site-percolation on the triangular lattice [Smi01], the level lines of the discrete Gaussian free field [SS09], the interfaces of the random cluster model associated with the Ising model [Smi07].

A start to taking limits as above is to define the natural parametrization for SLE . Possible definitions and constructions for a natural parametrization were suggested in [LS09]. As well as giving conjectures, one definition was proposed in terms of the Doob-Meyer decomposition of a path. We review this construction in Section 1.3. Although they conjectured that this definition is valid for all $\kappa < 8$, they were only able to establish the result for $\kappa < \kappa_0 = 4(7 - \sqrt{33})$. The technical problem came from difficult second moment estimates for the reverse Loewner flow.

In this paper, we prove that the definition in [LS09] is valid for $\kappa < 8$. Instead of using the reverse Loewner flow, we use a difficult estimate of Beffara [Bef08] on the forward Loewner flow to establish the necessary uniform integrability to apply the Doob-Meyer theorem. Beffara's estimate was the key step in his proof of the Hausdorff dimension of SLE_κ curves. This estimate has recently been improved [LW10] and used to establish a multi-point Green's function for SLE_κ . We use this Green's function to give an appropriate two-interior-point local martingale. By establishing a correlation inequality for this Green's function, we are able to give a relatively simple proof of the existence of the natural parametrization.

1.2 Notation

In this subsection we set up the notation for SLE_κ . For more background, see for example, [Wer04, GK04, Car05, Law05, Law09a].

Throughout this paper we let $\kappa < 8$ and $a = 2/\kappa > 1/4$, and we allow all constants to depend on κ . We let $d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a}$ be the Hausdorff dimension of the paths. We parametrize the maps so that

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad (1)$$

where $U_t = -B_t$ is a standard Brownian motion. It can be shown [RS05, LSW04] that a.s. g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ for every $t \geq 0$ and $\gamma(t) := g_t^{-1}(U_t)$ is a continuous curve which is the SLE path. The domain of definition of g_t is the unbounded connected component H_t

of $\mathbb{H} \setminus \gamma[0, t]$. We shall denote by K_t the closure of the complement of H_t in \mathbb{H} . If $z \in \overline{\mathbb{H}} \setminus \{0\}$, let

$$Z_t(z) = X_t(z) + iY_t(z) = g_t(z) + B_t.$$

Then the Loewner equation can be written as

$$dZ_t(z) = \frac{a}{Z_t(z)} dt + dB_t.$$

$$dX_t(z) = \frac{aX_t(z)}{|Z_t(z)|^2} dt + dB_t, \quad \partial_t Y_t(z) = -\frac{aY_t(z)}{|Z_t(z)|^2}.$$

The Loewner equation is valid up to the time

$$T_z = \sup\{t : Y_t(z) > 0\}.$$

Let

$$\Upsilon_t(z) = \frac{Y_t(z)}{|g'_t(z)|}, \quad \theta_t(z) = \arg Z_t(z), \quad S_t(z) = \sin \theta_t(z) = \frac{Y_t(z)}{|Z_t(z)|}.$$

It is not difficult to see that $\Upsilon_t(z)$ is $1/2$ times the conformal radius of H_t with respect to z , by which we mean that if $f : \mathbb{D} \rightarrow H_t$ is a conformal transformation with $f(0) = z$, then $|f'(0)| = 2\Upsilon_t(z)$. Using the Schwarz lemma and the Koebe $(1/4)$ -theorem we can see that

$$\Upsilon_t(z) \asymp_2 \text{dist}(z, \mathbb{R} \cup \gamma(0, t]),$$

where \asymp_2 means that both sides are bounded above by 2 times the other side. Using the Loewner equation (1) we see that

$$\partial_t \Upsilon_t(z) = -\Upsilon_t(z) \frac{2aY_t(z)^2}{|Z_t(z)|^4}. \quad (2)$$

In particular, $\Upsilon_t(z)$ decreases with t and hence we can define

$$\Upsilon(z) = \lim_{t \rightarrow T_z^-} \Upsilon_t(z)$$

which satisfies

$$\Upsilon(z) \asymp_2 \text{dist}[z, \mathbb{R} \cup \gamma(0, \infty)].$$

Using Itô's formula, we can see that

$$d\theta_t(z) = \frac{(1-2a)X_t(z)Y_t(z)}{|Z_t(z)|^4} dt - \frac{Y_t(z)}{|Z_t(z)|^2} dB_t. \quad (3)$$

The *Green's function* (for SLE_κ from 0 to ∞ in \mathbb{H}) is defined by

$$G(z) = y^{d-2} [\sin \arg z]^{4a-1} = y^{\frac{1}{4a}+4a-2} |z|^{1-4a},$$

where $z = x + iy = |z|e^{i\theta}$. Itô's formula shows that

$$M_t(z) = |g'_t(z)|^{2-d} G(Z_t(z)) = \Upsilon_t(z)^{d-2} S_t(z)^{4a-1}, \quad t < T_z, \quad (4)$$

is a local martingale satisfying

$$dM_t(z) = \frac{(1-4a)X_t(z)}{|Z_t(z)|^2} M_t(z) dB_t.$$

More generally, if D is a simply connected domain, $z \in D$, and w_1, w_2 are distinct points in ∂D , we can define $\Upsilon_D(z), S_D(z; w_1, w_2)$ using the following scaling rules: if $f : D \rightarrow f(D)$ is a conformal transformation, then

$$\Upsilon_{f(D)}(f(z)) = |f'(z)| \Upsilon_D(z), \quad S_{f(D)}(f(z); f(w_1), f(w_2)) = S_D(z; w_1, w_2).$$

The Green's function $G_D(z; w_1, w_2)$ is defined by

$$G_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1},$$

and satisfies the scaling rule

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)). \quad (5)$$

Under this definition, the local martingale in (4) can be rewritten as

$$M_t(z) = G_{H_t}(z; \gamma(t), \infty).$$

The following easy lemma is useful for estimating $S_D(z; w_1, w_2)$.

Lemma 1.1. *Suppose D is a simply connected domain, $w_1, w_2 \in \partial D$, and $f : \mathbb{H} \rightarrow D$ is a conformal transformation with $f(0) = w_1, f(\infty) = w_2$. Let $\partial_+ = f[(0, \infty)], \partial_- = f[(-\infty, 0)]$. If $z \in D$, let*

$$q = q_D(z; w_1, w_2) = \min \{h_D(z, \partial_+), h_D(z, \partial_-)\},$$

where h_D denotes harmonic measure. Then

$$2q \leq S_D(z; w_1, w_2) \leq \pi q. \quad (6)$$

Proof. We first note that q, h_D, S_D are conformal invariants, so it suffices to prove the result for $D = \mathbb{H}, w_1 = 0, w_2 = \infty$ and by symmetry we may assume that $\theta := \arg z \leq \pi/2$. By explicit calculation, we can see that $\theta = \pi q$, and hence (6) follows from the estimate

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1, \quad 0 < x \leq \frac{\pi}{2}.$$

□

Using Girsanov's theorem, it can be shown that

$$\lim_{\epsilon \rightarrow 0+} \epsilon^{d-2} \mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon\} = c_* G(z), \quad c_* = 2 \left[\int_0^\pi \sin^{4a} x \, dx \right]^{-1}.$$

A proof of this is given in [Law09a], but we include a self-contained proof in this paper (Proposition 2.3) which also estimates the error term. It follows that if D is a simply connected domain, $z \in D$; γ is an SLE_κ curve connecting distinct boundary points $w_1, w_2 \in \partial D$, and D_∞ denotes the component of $D \setminus \gamma$ containing z , then

$$\mathbb{P}\{\Upsilon_{D_\infty}(z) \leq \epsilon\} \sim c_* \epsilon^{2-d} G_D(z; w_1, w_2), \quad \epsilon \rightarrow 0+.$$

1.3 Review of natural parametrization

We will briefly review the construction in [LS09]. The starting point is the following proposition.

Proposition A. ([LS09]) *Suppose that there exists a parametrization for SLE_κ in \mathbb{H} satisfying the domain Markov property and the conformal invariance assumption. For a fixed Lebesgue measurable subset $S \subset \mathbb{H}$, let $\Theta_t(S)$ denote the process that gives the amount of time in this parametrization spent in S before time t (in the half-plane capacity parametrization), and suppose further that $\Theta_t(S)$ is \mathcal{F}_t adapted for all such S . If $\mathbb{E}\Theta_\infty(D)$ is finite for all bounded domains D , then it must be the case that (up to multiplicative constant)*

$$\mathbb{E}\Theta_\infty(D) = \int_D G(z) \, dA(z),$$

where dA denotes integration with respect to area, and more generally,

$$\mathbb{E}[\Theta_\infty(D) - \Theta_t(D) | \mathcal{F}_t] = \int_D M_t(z) \, dA(z).$$

Let \mathcal{D} denote the set of bounded domains $D \subset \mathbb{H}$ with $\text{dist}(\mathbb{R}, D) > 0$. Write

$$\mathcal{D} = \bigcup_{m=1}^{\infty} \mathcal{D}_m,$$

where \mathcal{D}_m denotes the set of domains D with

$$D \subset \{x + iy : |x| < m, 1/m < y < m\}.$$

For any process $\Theta_t(D)$ with finite expectations, by Proposition A., one has

$$\Psi_t(D) = \mathbb{E}[\Theta_\infty(D) | \mathcal{F}_t] - \Theta_t(D), \tag{7}$$

where $\Psi_t(D) := \int_D M_t(z) \, dA(z)$, which is a supermartingale in t because $M_t(z)$ is a non-negative local martingale. It is also not difficult to prove that $\Psi_t(D)$ is in fact continuous

as a function of t by its definition. Assuming the conclusion of Proposition A, the first term on the right-hand side of (7) is a martingale and the map $t \mapsto \Theta_t(D)$ is increasing. Inspired by the continuous case of the standard Doob-Meyer theorem [DM82]: any continuous supermartingale can be written uniquely as the sum of a continuous adapted decreasing process with initial value zero and a continuous local martingale, Lawler and Shieffield in [LS09] introduce the following definition.

Definition ([LS09])

- If $D \in \mathcal{D}$, then the natural parametrization $\Theta_t(D)$ is the unique continuous, increasing process such that

$$\Psi_t(D) + \Theta_t(D)$$

is a martingale (assuming such a process exists).

- If $\Theta_t(D)$ exists for each $D \in \mathcal{D}$, the natural parametrization in \mathbb{H} is given by

$$\Theta_t = \lim_{m \rightarrow \infty} \Theta_t(D_m),$$

where $D_m = \{x + iy : |x| < m, 1/m < y < m\}$.

The main result in that paper is the following.

Theorem B. ([LS09]) *If $\kappa < \kappa_0 := 4(7 - \sqrt{33})$, there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D) = 0$ such that*

$$\Psi_t(D) + \Theta_t(D)$$

is a martingale. Moreover, with probability one for all t

$$\Theta_t(D) = \lim_{n \rightarrow \infty} \sum_{j \leq t2^n} \int_{\mathbb{H}} |\hat{f}'_{\frac{j-1}{2^n}}(z)|^d \phi(z2^{n/2}) G(z) 1\{\hat{f}_{\frac{j-1}{2^n}}(z) \in D\} dA(z), \quad (8)$$

where $\phi(z)$ is defined by $\mathbb{E}[M_1(z)] = M_0(z)(1 - \phi(z))$, and $\hat{f}_s(z) = g_s^{-1}(z + U_s)$.

As for the proof, they start by discretizing time and finding an approximation for $\Theta_t(D)$. This time discretization is the first step in proving the Doob-Meyer decomposition for any supermartingale. The second step is to take the limit. For this purpose, they use the reverse-time flow for the Loewner equation to derive uniform second moment estimates for the approximations when $\kappa < \kappa_0$. This estimate is the most difficult one in all their derivation. Then they can take a limit both in L^2 and with probability one.

1.4 Multi-point Green's function

As the main step in proving the Hausdorff dimension of the SLE curve, Beffara [Bef08] proved the following lemma.

Lemma 1.2. *Suppose D is a bounded subdomain of \mathbb{H} with $\text{dist}(D, \mathbb{R}) > 0$. Then there exists $c_D < \infty$ such that if $z, w \in D$ and $\epsilon > 0$,*

$$\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \epsilon\} \leq c_D \epsilon^{2(2-d)} |z - w|^{d-2}.$$

Recently, Lawler and Werner [LW10] extended Beffara's argument to show that

$$\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \delta\} \leq c_D \epsilon^{2-d} \delta^{2-d} |z - w|^{d-2}. \quad (9)$$

Building on this, they show that there is a multi-point Green's function $G(z, w)$ such that

$$\lim_{\epsilon, \delta \rightarrow 0+} \epsilon^{d-2} \delta^{d-2} \mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \delta\} = c_*^2 G(z, w). \quad (10)$$

Although a closed form of the function $G(z, w)$ is not given, it is shown that

$$G(z, w) = G(z) G(w) [F(z, w) + F(w, z)] \quad (11)$$

where

$$F(z, w) = \frac{\mathbb{E}_z^* [|g'_T(w)|^{2-d} G(Z_T(w))]}{G(w)},$$

\mathbb{E}_z^* denotes expectation with respect to two-sided radial SLE_κ through z (see Section 2 for definitions) and $T = T_z = \inf\{t : \gamma(t) = z\}$. Roughly speaking, $G(z, w)$ represents the probability of going through z and w , and $G(z) G(w) F(z, w)$ represents the probability of going first through z and then through w . Using (5), we can write

$$G(w) F(z, w) = \mathbb{E}_z^* [G_{D_T}(w; z, \infty)].$$

From the definition, we can see that if $r > 0$,

$$G(z, w) = r^{2(2-d)} G(rz, rw), \quad F(z, w) = F(rz, rw).$$

More generally, if D is a simply connected domain with boundary points z_1, z_2 , we can define $F_D(z, w; z_1, z_2)$ by conformal invariance.

Let

$$M_t(z, w) = |g'_t(z)|^{2-d} |g'_t(w)|^{2-d} G(Z_t(z), Z_t(w)) = \frac{M_t(z) M_t(w) G(Z_t(z), Z_t(w))}{G(Z_t(z)) G(Z_t(w))}. \quad (12)$$

This is the so-called two-interior-point local martingale. A similar two-boundary-point local martingale appears in [SZ10]. Using (10), we can see if $\epsilon > 0$ and T_ϵ is the first time such

that $\Upsilon_{T_\epsilon}(z) \geq \epsilon$ or $\Upsilon_{T_\epsilon}(w) \geq \epsilon$, then $\mathbb{E}[M_{T_\epsilon}(z, w)] = M_0(z, w) = G(z, w)$. Hence, by Fatou's lemma, for every stopping time T ,

$$G(z, w) \geq \mathbb{E}[M_T(z, w)]. \quad (13)$$

For our main result we will need the following two estimates about $G(z, w)$. The first follows immediately from (9) and (10); establishing the second is the main technical work in this paper.

Lemma 1.3.

- Suppose D is a bounded subdomain of \mathbb{H} with $\text{dist}(D, \mathbb{R}) > 0$. Then there exists $c_D < \infty$ such that if $z, w \in D$,

$$G(z, w) \leq c_D |z - w|^{d-2}.$$

- There exists $c > 0$ such that for all $z, w \in \mathbb{H}$,

$$G(z, w) \geq c G(z) G(w). \quad (14)$$

Note that (14) is equivalent to saying that there exists c such that

$$F(z, w) + F(w, z) \geq c.$$

We remark that

$$\inf_{z, w} F(z, w) = 0.$$

Indeed, one can check that

$$\lim_{y \rightarrow \infty} F(iy, i/y) = 0.$$

The basic idea is that if y is large then the chance that the SLE path goes through yi and then through i/y is much smaller than the probability of going through i/y and then through iy .

Corollary 1.4. *If D is a bounded subdomain of \mathbb{H} with $\text{dist}(D, \mathbb{R}) > 0$, then there exists $c_D < \infty$ such that if $z, w \in D$, and T is a stopping time,*

$$\mathbb{E}[M_T(z) M_T(w)] \leq c_D |z - w|^{d-2}. \quad (15)$$

Proof. Using (12), (13), and Lemma 1.3, we get

$$\mathbb{E}[M_T(z) M_T(w)] \leq c \mathbb{E}[M_T(z, w)] \leq c G(z, w) \leq c_D |z - w|^{d-2}.$$

□

1.5 The main theorems

As in Lawler and Sheffield, we will prove that there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D) = 0$ such that $\Psi_t(D) + \Theta_t(D)$ is a martingale. The basic idea of the proof is the same. Here, we show how (15) yields the uniform integrability (class \mathfrak{D}) needed to establish the existence of the martingale.

Theorem 1. *If $0 < \kappa < 8$, there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D) = 0$ such that*

$$\Psi_t(D) + \Theta_t(D)$$

is a martingale. Moreover, let

$$\Theta_{t,n}(D) = \sum_{j \leq t2^n} \int_{\mathbb{H}} |\hat{f}'_{\frac{j-1}{2^n}}(z)|^d \phi(z2^{n/2}) G(z) 1\{\hat{f}_{\frac{j-1}{2^n}}(z) \in D\} dA(z),$$

then for any stopping time T ,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \Theta_{T,n}(D) - \Theta_T(D) \right| \right] = 0.$$

Remark From Theorem B, we know that $\mathbb{E}M_1(z) = M_0(z)(1 - \phi(z)) < M_0(z)$. So $M_t(z)$ is a local martingale and a supermartingale which is not a proper martingale. This implies that $\Psi_t(D)$ is not a proper martingale too. Since the theorem establishes that $\Psi_t(D) + \Theta_t(D)$ is actually a martingale, $\Theta_t(D)$ is non-trivial. In other words, it is not identically zero.

Before discussing how to prove Theorem 1, let us briefly review Doob-Meyer decomposition for supermartingales of class \mathfrak{D} . First, we assume that Ψ_t is a supermartingale with respect to a filtration \mathcal{F}_t , defined on the interval $[0, \infty)$. We also suppose that \mathcal{F}_t satisfies the usual conditions. The supermartingale $\{\Psi_t, \mathcal{F}_t, t \geq 0\}$ is said to be of *class \mathfrak{D}* if the family $\{\Psi_T : T \text{ is an almost surely finite stopping time}\}$ is uniformly integrable. The following result is about Doob-Meyer decomposition for supermartingales of class \mathfrak{D} . One can refer to Section 1.4 of [KS91] for more details.

Theorem C. [Doob-Meyer Decomposition] ([KS91]) *Let $\{\Psi_t, \mathcal{F}_t, t \geq 0\}$ be a continuous supermartingale of class \mathfrak{D} . Then there exists a continuous predictable, non-decreasing process $\{A_t, \mathcal{F}_t, t \geq 0\}$, such that $A_0 = 0$, A_∞ is integrable, and*

$$\Psi_t = \mathcal{M}_t - A_t.$$

where $\mathcal{M}_t := \mathbb{E}[A_\infty + \Psi_\infty | \mathcal{F}_t]$ is a martingale. This decomposition is unique up to indistinguishability, i.e. if $\{\mathcal{M}'_t, \mathcal{F}_t, t \geq 0\}$ and $\{A'_t, \mathcal{F}_t, t \geq 0\}$ are a martingale and a predictable, non-decreasing process satisfying the above properties respectively, then

$$\mathbb{P}\{\mathcal{M}_t = \mathcal{M}'_t, A_t = A'_t, \forall t \geq 0\} = 1.$$

In order to prove Theorem 1, we need one result from Section 20 of Chapter VII in [DM82] about Doob-Meyer Decompositions of an increasing sequence of positive supermartingales.

Theorem D. ([DM82]). Let $\{\Psi_t^n, \mathcal{F}_t, t \geq 0\}$ be an sequence of positive supermartingales, whose limit $\{\Psi_t, \mathcal{F}_t, t \geq 0\}$ belongs to class \mathfrak{D} and is regular. Let A_t^n and A denote the non-decreasing processes associated with Ψ_t^n and Ψ_t , respectively. Then for any stopping time T ,

$$\lim_{n \rightarrow \infty} \mathbb{E}[|\Psi_T^n - \Psi_T|] = 0.$$

Proof of Theorem 1 given Lemma 1.3. From Theorem C, in order to show the existence of $\Theta_t(D)$ with the desired properties, we need to find a $c(D) < \infty$ such that for every stopping time T ,

$$\mathbb{E}[\Psi_T^2(D)] \leq c(D). \quad (16)$$

From (15), we have

$$\begin{aligned} \mathbb{E}[\Psi_T^2(D)] &= \int_D \int_D \mathbb{E}[M_T(z_1)M_T(z_2)] dA(z_1)dA(z_2) \\ &\leq c_D \int_D \int_D |z_1 - z_2|^{d-2} dA(z_1)dA(z_2) \leq c(D). \end{aligned}$$

This gives the first result. Now we turn to the second result. For all t we set

$$\Psi_t^n(D) = \mathbb{E}[\Psi_{(i+1)2^{-n}}(D)|\mathcal{F}_t] \quad \text{if } i2^{-n} \leq t < (i+1)2^{-n}.$$

Hence $\{\Psi_t^n(D), \mathcal{F}_t, t \geq 0\}$ is a positive supermartingale bounded above by $\Psi_t(D)$, which increases to $\Psi_t(D)$ as $n \rightarrow \infty$. Let A_t^n be the non-decreasing process associated with $\Psi_t^n(D)$. Then for $i2^{-n} \leq t < (i+1)2^{-n}$,

$$A_t^n = \sum_{0 \leq k < i} \mathbb{E}[\Psi_{k2^{-n}}(D) - \Psi_{(k+1)2^{-n}}(D)|\mathcal{F}_{k2^{-n}}].$$

Actually, by the change of variables and the scaling rule of ϕ , we have

$$A_t^n = \Theta_{t,n}(D).$$

By Theorem D we can complete the proof. □

Remark Although we have used the existence of the multi-point Green's function $G(z, w)$ as established in [LW10], we could have proven our result here without its existence. In fact, an earlier draft of our paper derived the theorem from Beffara's estimate replacing (14) with

$$\mathbb{P}\{\Upsilon(z) \leq \epsilon, \Upsilon(w) \leq \delta\} \geq c \mathbb{P}\{\Upsilon(z) \leq \epsilon\} \mathbb{P}\{\Upsilon(w) \leq \delta\}.$$

However, the argument is cleaner when written in terms of $G(z, w)$ so we use it here.

The next result shows that our natural parametrization could be a kind of conformal Minkowski measure (although is not the same definition as the conformal Minkowski content defined in [LS09]).

Theorem 2. *Let $\gamma^\epsilon(0, t] = \{z \in \mathbb{H} : \sup_{0 \leq s \leq t} M_s(z) \geq \epsilon^{2-d}\}$. If $0 < \kappa < 8$, then for any stopping time T ,*

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\left| \epsilon^{2-d} A(D \cap \gamma^\epsilon(0, T]) - \Theta_T(D) \right| \right] = 0.$$

Proof of Theorem 2 given Lemma 1.3. Note that

$$\epsilon^{2-d} A(D \cap \gamma^\epsilon(0, t]) = \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \in \gamma^\epsilon(0, t]\} dA(z),$$

where $\tilde{\tau}_\epsilon(z) = \inf\{t \geq 0 : M_t(z) \geq \epsilon^{d-2}\}$. The above integral appears in the identity

$$\begin{aligned} & \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \notin \gamma^\epsilon(0, t]\} dA(z) \\ &= \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) dA(z) - \int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \in \gamma^\epsilon(0, t]\} dA(z). \end{aligned}$$

This is actually the Doob-Meyer decomposition of the supermartingale $\int_D M_{t \wedge \tilde{\tau}_\epsilon(z)}(z) I\{z \notin \gamma^\epsilon(0, t]\} dA(z)$. By Theorem D and (16), we can complete the proof. \square

1.6 Outline of the paper

In Section 2, we will develop the theory of two-sided radial SLE in order to prove Proposition 3.1. Although this was discussed somewhat in [Law09a], our treatment here is self-contained. One main goal is the refined “one-point” estimate in Proposition 2.3. The first step of the proof goes back to [RS05] in which the radial parametrization was used to establish a weaker form of the estimate. Here we use Girsanov’s theorem to reduce the question to the rate of convergence to equilibrium of a simple one-dimensional diffusion. The sharp estimate is used in [LW10] to improve Beffara’s estimate and show existence of the multi-point Green’s function. When considering radial SLE_κ going through points $z = x + iy$ with $|x| \gg y$, it is useful to compare this to a process headed to x . We call the appropriate one-dimensional process “two-sided chordal SLE_κ ” and give some of its properties.

The only thing we need to prove Theorem 1 is the estimate (14). We do not know the optimal value for c , but we can guarantee that our proof does *not* give the optimal value. In Section 3.1, we spend a relatively long time deriving an estimate that states roughly that there is a positive probability that a two-sided radial path stays with a small distance of an “ L ”-shape. The proof uses two well-known ideas:

- If $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function with $f(0) = 0$ and $\epsilon > 0$, and U_t is a standard Brownian motion, then the probability that $\|f - U\|_\infty < \epsilon$ is positive, where $\|\cdot\|_\infty$ denotes the supremum norm.

- If the driving functions (using the Loewner equation) of two curves are close in the supremum norm, then the curves are close in the Hausdorff metric. This follows in a straightforward manner from the Loewner equation. (It is possible that the curves are not close in the supremum norm on curves, but this is not important for us).

It takes a little time to write out the details because our driving functions are those for two-sided radial and we need some uniformity in the estimates. Readers who are convinced that this can be done can skip the proof of Proposition 3.1. Proposition 3.2 is a simple deterministic estimate which uses Proposition 3.1 to establish (14) for z, w far apart.

It remains to prove (14) for z, w close and this is the goal of Section 3.2. Indeed, this does not seem that it should be difficult, since the events of getting close to z and getting close to w should be very positively correlated!. We give the argument in two cases: when $|z - w|$ is much smaller than $\text{Im}(z)$ and when z, w are close to each other and also close to the boundary.

2 Two-sided radial SLE

One can see that with probability one $M_{T_z-}(z) = 0$; indeed, with probability one $\Upsilon(z) > 0$ and $S_{T_z-}(z) = 0$. This local martingale blows up on the event of probability zero that $z \in \gamma(0, \infty)$. To be more precise, let

$$\tau_\epsilon(z) = \inf\{t : \Upsilon_t(z) \leq \epsilon\}.$$

Then $M_{t \wedge \tau_\epsilon(z)}(z)$ is a martingale. For convenience, we will fix z and write $M_t, X_t, Y_t, \tau_\epsilon, \dots$ for $M_t(z), X_t(z), Y_t(z), \tau_\epsilon(z), \dots$.

Two-sided radial SLE_κ through z is the process obtained from the Girsanov transformation by weighting by the local martingale M_t . We should really call this two-sided radial SLE_κ from 0 to ∞ going through z stopped when it reaches z , but we will just say two-sided radial SLE_κ through z . This could also be called SLE_κ conditioned to go through z (and stopped at T_z) although this is a conditioning on an event of probability zero. Using (4) and the Girsanov theorem, we see that for each ϵ

$$dB_t = \frac{(1 - 4a) X_t}{|Z_t|^2} dt + dW_t, \quad t \leq \tau_\epsilon, \quad (17)$$

where W_t is a standard Brownian motion in the weighted measure \mathbb{P}^* which can be defined by saying that if V is an event depending only on $\{B_t : 0 \leq t \leq \tau_\epsilon\}$,

$$\mathbb{P}^*(V) = \mathbb{P}_z^*(V) = G(z)^{-1} \mathbb{E}[M_{\tau_\epsilon} 1_V].$$

We write \mathbb{E}^* for expectations with respect to \mathbb{P}^* . There is an implicit z dependence in \mathbb{P}^* and \mathbb{E}^* ; when we need to make this explicit, we write $\mathbb{P}_z^*, \mathbb{E}_z^*$. It is not hard to show (see Section

2.1) that for every $\epsilon > 0$, $\mathbb{P}^*\{\tau_\epsilon < \infty\} = 1$. Since the SDE in (17) has no ϵ dependence, we can let $0 \leq t < T_z$. Note that

$$dX_t = \frac{(1 - 3a) X_t}{|Z_t|^2} dt + dW_t.$$

$$-U_t = B_t = X_t - a \int_0^t \frac{X_s}{|Z_s|^2} ds.$$

2.1 Radial parametrization

When studying the behavior of an SLE_κ curve near an interior point z , it is useful to reparametrize the curve so that $\log \Upsilon_t(z)$ decays linearly. This is the parametrization generally used for radial SLE and for this reason we call it the *radial parametrization*. We will study the radial parametrization in this subsection. We denote this time change as a function $\sigma(t)$ and we write $\hat{X}_t = X_{\sigma(t)}$, $\hat{Y}_t = Y_{\sigma(t)}$, etc.

We define σ by asserting that

$$\hat{\Upsilon}_t = \Upsilon_{\sigma(t)} = e^{-2at}.$$

Using (2), we see that

$$-2a\hat{\Upsilon}_t = \partial_t [\hat{\Upsilon}_t] = -\hat{\Upsilon}_t \frac{2a\hat{Y}_t^2}{|\hat{Z}_t|^4} [\partial_t \sigma(t)],$$

and hence

$$\partial_t \sigma(t) = \frac{|\hat{Z}_t|^4}{\hat{Y}_t^2}.$$

From (3), we see that $\hat{\theta}_t := \theta_{\sigma(t)}$ satisfies

$$d\hat{\theta}_t = 2a \cot \hat{\theta}_t dt + d\hat{W}_t, \tag{18}$$

where \hat{W}_t is a standard Brownian motion with respect to the weighted measure \mathbb{P}^* . Since $a > 1/4$, the process (18) stays in $(0, \pi)$ for all times. This implies that for all $\epsilon > 0$,

$$\mathbb{P}^*\{\tau_\epsilon < \infty\} = 1. \tag{19}$$

2.1.1 The corresponding SDE

The analysis of two-sided radial SLE relies on detailed properties of the equation (18) which fortunately are not very difficult to obtain. In this subsection we focus on this SDE and set $r = 2a > 1/2$. Constants in this section may depend on r . We consider the equation

$$dX_t = r \cot X_t dt + dB_t, \quad X_0 \in (0, \pi). \tag{20}$$

In studying this equation, it is useful to note that this is the equation that one gets if one starts with a Brownian motion X_t and then weights paths locally by $\sin^r X_t$. To be more precise, if X_t is a standard Brownian motion, then

$$M_t = [\sin X_t]^r \exp \left\{ -\frac{r(r-1)}{2} \int_0^t \frac{ds}{\sin^2 X_s} + \frac{1}{2} r^2 t \right\}$$

is a local martingale satisfying

$$dM_t = r [\cot X_t] M_t dX_t.$$

If we weight by the local martingale, then X_t satisfies (20) where B_t is a Brownian motion in the weighted measure. In fact, since the weighted process stays in $(0, \pi)$, one can see that M_t is actually a martingale.

The equation (20) is closely related to the Bessel equation

$$dX_t = \frac{r}{X_t} dt + dB_t. \quad (21)$$

This equation is obtained by starting with a Brownian motion X_t and weighting locally by X_t^r , i.e., by the martingale

$$N_t = X_t^r \exp \left\{ -\frac{r(r-1)}{2} \int_0^t \frac{ds}{X_s^2} \right\},$$

which satisfies

$$dN_t = r \frac{1}{X_t} N_t dX_t.$$

The next lemma gives a precise statement about the relationship.

Lemma 2.1. *There exists $c < \infty$ such that the following is true. Suppose μ_1 is the probability measure on paths given by*

$$X_t, \quad 0 \leq t \leq 1 \wedge T,$$

where X_t satisfies (20) with $X_0 = x$ and

$$T = \inf\{t : X_t \geq \pi/2\}.$$

Let μ_2 be the analogous probability measure using the equation (21). Then

$$\frac{1}{c} \leq \frac{d\mu_1}{d\mu_2} \leq c. \quad (22)$$

Proof. From the explicit forms of M_t, N_t , one can see there exists c such that

$$c^{-1} N_t \leq M_t \leq c N_t, \quad 0 \leq t \leq T \wedge 1.$$

□

By comparison with the Bessel process we see that the process satisfying (20) never leaves $(0, \pi)$. The invariant density for the process is

$$h(x) = C_{2r} \sin^{2r} x, \quad C_{2r}^{-1} = \int_0^\pi \sin^{2r} y \, dy.$$

(This can be derived in a number of ways. It essentially follows because the invariant density for Brownian motion weighted by a function F is proportional to F^2 .) In particular,

$$\int_0^\pi h(x) [\sin x]^{1-2r} \, dx = 2 C_{2r}. \quad (23)$$

If $t \geq 0, x \in (0, \pi)$ we define

$$\psi(t, x) = \mathbb{E}^x[(\sin X_t)^{1-2r}],$$

where X_t satisfies (20). Note that $\psi(t, x) = \psi(t, \pi - x)$. Let

$$\tilde{\psi}(t, x) = \mathbb{E}^x[X_t^{1-2r}],$$

where X_t satisfies (21). Using the previous lemma, we can see there exist c_1, c_2 such that

$$c_1 \psi(t, x) \leq \tilde{\psi}(t, x) \leq c_2 \psi(t, x), \quad 0 < x \leq \pi/2, \quad 0 \leq t \leq 1. \quad (24)$$

Let

$$v(t, x) = [x \vee \sqrt{t \wedge 1}]^{1-4a}.$$

The next lemma collects the facts about the SDE that we will need.

Lemma 2.2. *There exists $c < \infty$ such that the following is true for all $x \in (0, \pi/2]$.*

- If $t \geq 1$,

$$|\psi(t, x) - 2C_{2r}| \leq c e^{-(r+\frac{1}{2})t}. \quad (25)$$

- If $t \geq 0$,

$$c^{-1} v(t, x) \leq \psi(t, x) \leq c v(t, x). \quad (26)$$

- For every $\epsilon > 0$, there is a $\delta > 0$ such that if $t > 0, 0 < x < \pi$, and $I \subset \mathbb{R}$ with

$$\mathbb{P}^x\{X_t \in I\} \geq \epsilon,$$

then

$$\mathbb{E}^x[(\sin X_t)^{1-2r}; X_t \in I] \geq \delta v(t, x). \quad (27)$$

Proof. The fact that

$$\lim_{t \rightarrow \infty} \psi(t, x) = 2C_{2r}$$

follows from (23). To get the error estimate, one needs the next eigenvalue. We will pull it out of the hat. Suppose $x_0 \leq \pi/2$ and let T be the first time that the process reaches $\pi/2$. A simple application of Itô's formula shows that

$$M_t = \cos(X_t) e^{(r+\frac{1}{2})t},$$

is a martingale. Using this one can show that

$$\mathbb{P}^{x_0}\{T \geq t\} \leq c e^{-(r+\frac{1}{2})t},$$

with a constant independent of the starting point and then a coupling argument can be used to show that if we start two processes, one at x_0 and the other in the invariant density, then the probability that they do not couple by time t is $O(e^{-(r+\frac{1}{2})t})$.

Note that (26) for $t \geq 1$ follows immediately from (25), so it suffices to prove it for $t \leq 1$.

A coupling argument shows that if $x \leq x_1$, then $\mathbb{P}^x\{X_t \leq y\} \geq \mathbb{P}^{x_1}\{X_t \leq y\}$. Therefore, if $X_0 = x$ and $h = C_{2r}(\sin x)^{2r}$ denotes the invariant density,

$$\mathbb{P}^x\{X_t \leq y\} \leq \frac{\int_0^x \mathbb{P}^s\{X_t \leq y\} h(s) ds}{\int_0^x h(s) ds} \leq \frac{\int_0^y h(s) ds}{\int_0^x h(s) ds} \leq c (y/x)^{2r+1}.$$

By integrating, we see for all t ,

$$\mathbb{E}^x[(\sin X_t)^{1-2r}] \leq c x^{1-2r} + \mathbb{E}^x[(\sin X_t)^{1-2r}; X_t \leq x] \leq c x^{1-2r}.$$

This gives the upper bound for $t \leq x^2$; for the lower bound for these t , we need only note that there is a positive probability that the process starting at x stays within distance $x/2$ of x up to time x^2 .

For the remainder we consider $x^2 \leq t \leq 1$. The estimates are more easily done for the Bessel process using scaling. Since $t \leq 1$, it suffices by (24) to prove that

$$\tilde{\psi}(t, x) \asymp t^{\frac{1}{2}-r}, \quad x^2 \leq t \leq 1, \quad 0 < x \leq \pi/2,$$

where the implicit constants in the \asymp notation are uniform over t, x . Suppose X_t satisfies (21) with $X_0 = x$. Then X_t has the same distribution as $x Y_{t/x^2}$, where Y_t satisfies (21) with $Y_0 = 1$. We consider Y_t for $1 \leq t \leq x^{-2}$.

To study this equation it is convenient to consider $Z_t = e^{-t/2} Y_{e^t}$. Note that

$$\begin{aligned} dZ_t &= \left[-e^{-t/2} Y_{e^t} + e^{-t/2} \frac{r e^t}{Y_{e^t}} \right] dt + e^{-t/2} dB_{e^t/2} \\ &= \left[-Z_t + \frac{r}{Z_t} \right] dt + d\hat{B}_t, \end{aligned}$$

where \hat{B}_t is a standard Brownian motion. This is a positive recurrent SDE with invariant density proportional to

$$x^{2r} e^{-x^2}.$$

We consider this with the initial condition $Z_{-\infty} = \infty$. (One can show this makes sense by writing the equation for $R_t = 1/Z_t$ and showing that this can be well defined with $R_{-\infty} = 0$.) By time 0, the process is within a constant multiple of the invariant density. All we will need from this are the following easily derived facts.

$$\mathbb{E} [Z_t^{1-2r}] \leq c, \quad t \geq 0.$$

Secondly, for every $\epsilon > 0$, there is a $\delta > 0$ such that if $t \geq 0$ and

$$\mathbb{P}\{Z_t \geq K\} \geq \epsilon,$$

then

$$\mathbb{E} [Z_t^{1-2r}; Z_t \geq K] \geq \delta.$$

□

2.1.2 The one-point estimate

The results of the previous subsection will be used with $r = 2a > 1/2$. We use the radial parametrization to prove the next proposition.

Proposition 2.3. *If $z = x + iy$ and $\theta = \arg z$, then for $\epsilon \leq y$,*

$$\mathbb{P}\{\tau_\epsilon < \infty\} = \epsilon^{2-d} G(z) \psi(t, \theta),$$

where t satisfies $\epsilon = y e^{-2ta}$, i.e.,

$$\mathbb{P}\{\tau_\epsilon < \infty\} = \epsilon^{2-d} G(z) \psi\left(\frac{\log(y/\epsilon)}{2a}, \theta\right).$$

It follows that for $1/2 \leq \epsilon \leq 1$,

$$\mathbb{P}\{\tau_{\epsilon y} < \infty\} \asymp \theta^{4a-1} [\theta \vee \sqrt{1-\epsilon}]^{1-4a} = \min\{1, (\theta/\sqrt{1-\epsilon})^{4a-1}\}.$$

For $\epsilon \leq 1/2$,

$$\mathbb{P}\{\tau_{\epsilon y} < \infty\} \asymp \epsilon^{2-d} \theta^{4a-1}.$$

Proof. Let $\tau = \tau_\epsilon$. Since $M_{t \wedge \tau}$ is a martingale, we have

$$G(z) = \mathbb{E} [M_{t \wedge \tau}] = \mathbb{E} [M_t; t < \tau] + \mathbb{E} [M_\tau; \tau \leq t].$$

We can let $t \rightarrow \infty$, and from (19) and the dominated convergence theorem, we can deduce that

$$G(z) = \mathbb{E} [M_\tau; \tau < \infty].$$

Also,

$$\mathbb{P}\{\tau < \infty\} = \epsilon^{2-d} \mathbb{E} [\Upsilon_\tau^{d-2}; \tau < \infty] = \epsilon^{2-d} \mathbb{E} [M_\tau S_\tau^{1-4a}; \tau < \infty] = \epsilon^{2-d} G(z) \mathbb{E}^* [S_\tau^{1-4a}].$$

From Girsanov, we see that

$$\mathbb{E}^* [S_\tau^{1-4a}] = \psi(t, \theta).$$

□

The following is a corollary of this and Lemma 2.2.

Proposition 2.4. *For $\epsilon \leq y$,*

$$\mathbb{P}\{\tau_\epsilon < \infty\} = c_* G(z) \epsilon^{2-d} \left[1 + O\left([\epsilon/y]^{1+\frac{1}{4a}}\right) \right]. \quad (28)$$

In particular, there exists $c < \infty$ such that for all z and all ϵ ,

$$\mathbb{P}\{\tau_\epsilon < \infty\} \leq c [\epsilon/\text{Im}(z)]^{2-d}. \quad (29)$$

It follows that for all z, ϵ ,

$$\mathbb{P}\{\tau_\epsilon < \infty\} \geq c G(z) \epsilon^{2-d}. \quad (30)$$

2.2 Two-sided chordal SLE_κ

When $z = x + iy$ with $|x| \gg y$, then the radial parametrization is not so useful, because the path can get close to z without significantly decreasing the radial parametrization. Fortunately, one can study such processes by studying another process which we call *two-sided chordal SLE_κ through x* . It is SLE_κ weighted locally by X_t^{1-4a} (as compared to $|Z_t|^{1-4a}$ for two-sided radial).

Let us fix $z = x + iy$ and for convenience we assume $x > 0$. We write $\tilde{X}_t = Z_t(x) = g_t(x) - U_t$ which satisfies

$$d\tilde{X}_t = \frac{a}{\tilde{X}_t} dt + dB_t.$$

Itô's formula shows that

$$d\tilde{X}_t^{1-4a} = X_t^{1-4a} \left[\frac{a(4a-1)}{X_t^2} dt + \frac{1-4a}{X_t} dB_t \right],$$

and hence

$$N_t = \tilde{X}_t^{1-4a} \exp \left\{ -a(4a-1) \int_0^t \frac{ds}{\tilde{X}_s^2} \right\} = \tilde{X}_t^{1-4a} g'_t(x)^{4a-1},$$

is a local martingale satisfying

$$dN_t = \frac{1-4a}{\tilde{X}_t} N_t dB_t.$$

Hence,

$$dB_t = \frac{1-4a}{\tilde{X}_t} dt + d\tilde{W}_t,$$

where \tilde{W} is a Brownian motion in the weighted measure. In particular,

$$d\tilde{X}_t = \frac{1-3a}{\tilde{X}_t} dt + d\tilde{W}_t.$$

From this we see that the terminal time T_x is finite with probability one in the weighted measure. Proposition 2.6 shows that $\gamma(T_x) = x$. (The proof is similar to the proof that SLE_κ hits points for $\kappa \geq 8$.) We precede this with a standard lemma about Bessel process whose proof we omit.

Lemma 2.5. *Suppose $r < 1/2$ and X_t satisfies the Bessel equation*

$$dX_t = \frac{r}{X_t} dt + dB_t, \quad X_0 = 1.$$

Let $T = \inf\{t : X_t = 0\}$. Then with probability one,

$$T < \infty, \quad \int_0^T \frac{dt}{X_t} < \infty, \quad \int_0^T \frac{dt}{X_t^2} = \infty.$$

Moreover, for every $\delta > 0$,

$$\mathbb{P} \left\{ 0 \leq X_t \leq 1 + \delta, \ 0 \leq t \leq \delta; \ T \leq \delta; \ \int_0^T \frac{dt}{X_t} \leq \delta \right\} > 0. \quad (31)$$

If $B_0 = 0$, then

$$-B_t = X_0 - X_t + \int_0^t \frac{r}{X_s} ds = 1 - X_t + \int_0^t \frac{r}{X_s} ds$$

and hence on the event described in (31)

$$-\delta \leq -B_t \leq 1 + \delta, \quad 0 \leq t \leq T.$$

Proposition 2.6. *Suppose $0 < x < x_1$ and let $R_t = Z_t(x_1) = g_t(x_1) - U_t$. Then with probability one, if γ is two-sided chordal to x with terminal time $T = T_x$,*

$$R_T > 0.$$

Proof. Since $1 - 3a < \frac{1}{2}$, Lemma 2.5 implies that with probability one $T < \infty$ and

$$\int_0^T \frac{dt}{\tilde{X}_t} < \infty, \quad (32)$$

$$\int_0^T \frac{dt}{\tilde{X}_t^2} = \infty. \quad (33)$$

Let $q(x, y)$ denote the probability that $R_T > 0$ given $\tilde{X}_0 = x, R_0 = y$. By Bessel scaling, $q(x, y) = q(1, y/x)$. We claim that $q(1, r) \rightarrow 1$ as $r \rightarrow \infty$. Indeed, we have

$$\log R_T = \log[R_T - \tilde{X}_T] = \log(y - x) + \int_0^T \partial_t [\log(R_t - \tilde{X}_t)] dt = \log(y - x) - \int_0^T \frac{a dt}{\tilde{X}_t R_t}.$$

From this and (32), we can deduce this fact. From this and the strong Markov property, it follows that on the event

$$\sup_{0 \leq t < T} \frac{R_t - \tilde{X}_t}{\tilde{X}_t} = \infty, \quad (34)$$

we must have $R_T > 0$.

If

$$L_t = \log \frac{R_t - \tilde{X}_t}{\tilde{X}_t},$$

then

$$dL_t = \left[-\frac{a}{\tilde{X}_t R_t} - \frac{\frac{1}{2} - 3a}{\tilde{X}_t^2} \right] dt - \frac{1}{\tilde{X}_t} d\tilde{W}_t.$$

Under a suitable time change $\hat{L}_t = L_{\sigma(t)}$, this can be written as

$$d\hat{L}_t = \left[-\frac{a\hat{X}_t}{\hat{R}_t} - \left(\frac{1}{2} - 3a \right) \right] dt + d\hat{B}_t,$$

for a standard Brownian motion \hat{B}_t . By (33), in this time change it takes infinite time for \hat{X}_t to reach zero, i.e., $\sigma(\infty) = T$. Since $\hat{X}_t \leq \hat{R}_t$, the drift term is bounded below by

$$-a - \left(\frac{1}{2} - 3a \right) > 0.$$

and hence $\hat{L}_t \rightarrow \infty$ as $t \rightarrow \infty$. In particular, (34) holds. \square

If $0 < x < x_1 < x_2$, then the random variable

$$\Delta = \Delta(x, x_1, x_2) = \max_{0 \leq t \leq T_x} \frac{Z_t(x_2)}{Z_t(x_1)}$$

is well defined and satisfies $1 < \Delta < \infty$. Moreover, Bessel scaling implies that the distribution of $\Delta(x, x_1, x_2)$ is the same as that of $\Delta(rx, rx_1, rx_2)$. The next lemma gives uniform bounds on the distribution in terms of $(x_2 - x)/(x_1 - x)$.

Lemma 2.7. *For every $\rho > 0$ there exists $C < \infty$ such that the following is true. Suppose $x < x_1 < x_2$ with $x_1 - x \geq \rho(x_2 - x)$. Then with probability at least $1 - \rho$, $\Delta(x, x_1, x_2) \leq C$.*

Proof. We fix ρ and allow constants to depend on ρ . By scaling and monotonicity, we may assume $x = 1$ and $x_1 - x = \rho(x_2 - x)$. Hence we write $x_1 = 1 + s\rho$, $x_2 = 1 + s$ where $s > 0$ (our estimates must be uniform over s). If s is very large, then since $T_x < \infty$, with probability one, one can find a K such that with probability at least $1 - \rho$ for all $\tilde{x} \geq K\rho$.

$$\frac{Z_0(\tilde{x})}{2} \leq Z_t(\tilde{x}) \leq 2Z_0(\tilde{x}), \quad 0 \leq t \leq T_x,$$

and hence with probability at least $1 - \rho$, $\Delta(1, 1 + s\rho, 1 + s) \leq 4$. For $1/2 \leq s \leq K$, we can bound

$$\Delta(1, 1 + s\rho, 1 + s) \leq \Delta\left(1, 1 + \frac{\rho}{2}, 1 + K\right).$$

For the remainder we assume that $s \leq 1/2$. Let

$$\xi = \inf\{t : |\gamma(t) - 1| = 2s\}.$$

By distortion estimates, we see that

$$g'_t(x) \asymp g'_t(x_1) \asymp g'_t(x_2), \quad 0 \leq t \leq \xi$$

with the implicit constants uniform over s . Since $g'_t(x')$ is an increasing function of x' , this implies

$$Z_t(x_1) \asymp Z_t(x_2), \quad 0 \leq t \leq \xi,$$

which gives uniform estimates (with probability one) on

$$\max_{0 \leq t \leq \xi} \frac{Z_t(x_2)}{Z_t(x_1)}$$

Also, using the 1/4-theorem with distortion estimates,

$$Z_\xi(x) \asymp Z_\xi(x_1) - Z_\xi(x) \asymp Z_\xi(x_2) - Z_\xi(x),$$

with again the estimates uniform over s . The conditional distribution of

$$\max_{\xi \leq t \leq T_x} \frac{Z_t(x_2)}{Z_t(x_1)}$$

given \mathcal{F}_ξ is the distribution of $\Delta(Z_\xi(x), Z_\xi(x_1), Z_\xi(x_2))$. Hence, we reduce this to the case where x, x_1, x_2 are comparable which we have already stated and handled. \square

Lemma 2.8. *For every $\rho > 0$, there is a $u > 0$ such that the following holds. Suppose $x > 0$ and γ is two-sided chordal SLE_κ to x . Let $0 < \epsilon \leq x$ and*

$$\xi = \inf\{t : |\gamma(t) - x| = \epsilon\}.$$

Define ψ by

$$\gamma(\xi) = x + \epsilon e^{i\psi}.$$

Then with probability at least $1 - \rho$, $\psi \geq u$.

Proof. As $\psi \rightarrow 0$, the extremal distance between $(x, x + \epsilon/2)$ and $(x + 2\epsilon, \infty)$ in H_ξ tends to ∞ . By conformal invariance of extremal distance, if ψ is very close to zero, then we can see that $Z_\xi(x + 2\epsilon) - Z_\xi(x) \gg Z_\xi(x + \frac{\epsilon}{2}) - Z_\xi(x)$. The previous lemma bounds the probability that this happens. \square

2.2.1 Comparison with two-sided radial

In this section, we fix $x > 0$ $\epsilon < x/2$. If $z \in \mathbb{H}$, we write $Z_t = Z_t(z) = X_t + iY_t$, $\tilde{X}_t = Z_t(x)$ as above. Recall that two-sided radial to z is the process obtained by weighting with respect to the local martingale

$$M_t = |Z_t|^{1-4a} (\Upsilon_t/y)^{\frac{1}{4a}-1} (Y_t/y)^{4a-1}.$$

Here we have included a constant $y^{2-4a-\frac{1}{4a}}$ to the usual local martingale, but this does not affect the weighting. Under this choice of M_t , $M_0 = |z|^{1-4a}$. Two-sided chordal is obtained by weighting by the local martingale

$$N_t = |\tilde{X}_t|^{1-4a} g'_t(x)^{4a-1}.$$

Let

$$\sigma = \inf\{t : |\gamma(t) - x| = 2\epsilon\}.$$

Distortion estimates and the 1/4-theorem imply that for $0 \leq t \leq \sigma$ and $|z - x| \leq \epsilon$,

$$g'_t(x) \asymp |g'_t(z)|, \quad \Upsilon_t \asymp \Upsilon_0 = y, \quad Y_t \asymp y |g'_t(z)|, \quad \tilde{X}_t \asymp |Z_t|$$

and hence

$$M_t \asymp N_t, \quad 0 \leq t \leq \sigma,$$

where the implicit constants are uniform over z . We have just proved the following.

Proposition 2.9. *There exists $c < \infty$, such that for every z , if μ denotes the measure on paths*

$$\gamma(t), \quad 0 \leq t \leq \sigma,$$

given by two-sided radial SLE_κ to $z = x + iy$ and ν denotes the analogous measure on paths using two-sided chordal to x , then

$$c^{-1} \leq \frac{d\mu}{d\nu} \leq c.$$

The following is a corollary of this proposition and Lemma 2.8.

Lemma 2.10. *For every $\rho > 0$, there is a $u > 0$ such that the following is true. Suppose $x > 0$ and $\delta < x$. Suppose $z \in \mathbb{H}$ with $|z - x| \leq \delta/2$. Let γ be a two-sided radial SLE_κ path through z , and let*

$$\sigma = \inf\{t : |\gamma(t) - x| = \delta\}.$$

Then,

$$\mathbb{P}_z^*\{S_\xi(z) \geq u\} \geq 1 - \rho.$$

3 Proof of (14)

3.1 Probability of L -shapes

If $z = x + iy$, let L_z denote the “ L ”-shape

$$L_z = [0, x] \cup [x, x + iy], \quad x \geq 0,$$

$$L_z = [x, 0] \cup [x, x + iy], \quad x \leq 0,$$

and if $\rho > 0$,

$$L_{z,\rho} = \{z' \in \overline{\mathbb{H}} : \text{dist}(z', L_z) \leq \rho|z|\}.$$

The goal of this section is to prove the following proposition.

Proposition 3.1. *For every $\rho > 0$, there is a $u > 0$ such that for all $z = x + iy \in \mathbb{H}$,*

$$\mathbb{P}_z^* \{ \gamma[0, T_z] \subset L_{z,\rho} \} \geq u.$$

The important thing is to show that we can choose u uniformly over z . Before discussing the proof of this proposition, let us show how it can be used to prove (14) for z and w sufficiently spread apart. We use the following deterministic estimate.

Proposition 3.2. *For every $0 < \rho \leq 1/4$ there exists $c < \infty$ such that the following holds. Suppose $z \in \mathbb{H}$, $w \in \mathbb{H} \setminus L_{z,2\rho}$, and suppose $\gamma : [0, T] \rightarrow \overline{\mathbb{H}}$ is a curve with $\gamma(0) = 0, \gamma(T) = z$ and $\gamma[0, T] \subset L_{z,\rho}$. Let $g = g_T$ be the corresponding conformal transformation and let $Z = g(w) - g(z)$. Then*

$$G(Z) \geq cG(w),$$

$$c^{-1} \leq |g'(w)| \leq c.$$

Proof. By scaling it suffices to consider $|z| = 1$ which we assume. The estimate is easy for $|w| \geq 2$ so we assume $w \in \mathbb{H} \setminus L_{z,2\rho}$ with $|w| \leq 2$. We fix ρ and allow (implicit or explicit) constants to depend on ρ . Let $J = \{w' : \text{Im}(w') \geq 3\}$. Using conformal invariance, one can see that $\text{Im}[g(w)]$ is comparable to the probability that a Brownian motion starting at w reaches J without hitting $\mathbb{R} \cup \gamma[0, T]$ and this is not smaller than the the probability of reaching J before hitting $\mathbb{R} \cup L_{z,\rho}$. It is not difficult to see that this is bounded below by a constant times $\text{Im}(w)$ and hence

$$c \text{Im}[w] \leq \text{Im}[g(w)] \leq \text{Im}[w].$$

Using (6), we also see that

$$S_{H_T}(w) \asymp \text{Im}[g(w)] \asymp \text{Im}[w] \asymp S_0(w)$$

(remember that the implicit constants depend on ρ). Also, $\text{dist}(w, \mathbb{R} \cup L_{z,\rho}) \asymp \text{Im}(w)$, and hence by the Koebe-1/4 theorem,

$$|g'(w)| \asymp 1.$$

Therefore, $G(Z) \asymp G(w)$. □

Corollary 3.3. *For every $\rho > 0$, there exists $c > 0$ such that if $z \in \mathbb{H}$ and $w \in \mathbb{H} \setminus L_{z,\rho}$, then $F(z, w) \geq c$.*

Proof. By Propositions 3.1 and 3.2, there exists $u = u(\rho) > 0$ such that

$$\mathbb{P}_z^* \{G_{H_T}(w; z, \infty) \geq u G(w)\} \geq u.$$

Therefore,

$$\mathbb{E}_z^* [G_{H_T}(w; z, \infty)] \geq u^2 G(w).$$

□

The proof of Proposition 3.1 combines a probabilistic estimate for the driving function and a deterministic estimate using the Loewner equation. Assume $z = x + iy$, $x \geq 0$, $y > 0$. Note that one obtains the hull $[x, x + iy]$ by solving (1) with $U_t \equiv x$, $0 \leq t \leq y^2/(2a)$. For every $\delta > 0$, let $E_{z,\delta}$ denote the event

$$E_{z,\delta} = \left\{ T_z \leq \frac{y^2}{2a} + \delta; \quad -\delta \leq U_t \leq x + \delta, 0 \leq t \leq \delta; \quad |U_t - x| \leq \delta, \delta \leq t \leq T_z \right\}.$$

Lemma 3.4. *For every $\rho > 0$, there exists $\delta > 0$ such that the following holds. Suppose $z = x + iy$ with $0 \leq x \leq 1$, $0 < y \leq 1$. Then on the event $E_{z,\delta}$,*

$$\gamma[0, T_z] \subset L_{z,\rho}.$$

Proof. This is a straightforward deterministic estimate using (1). One first shows that if δ is sufficiently small, then $|g_\delta(w) - w| \leq \rho/100$ for $\text{dist}(w, [0, x]) \geq \rho$. For $\delta \leq t \leq T_z$, we compare g_t to the corresponding function \tilde{g}_t obtained with $\tilde{U}_t \equiv x$. These estimates are standard (see, e.g., [Law05, Proposition 4.47]), and we omit the details. □

Therefore, Proposition 3.1 reduces to the following probabilistic estimate on the driving function.

Lemma 3.5. *For every $\delta > 0$, there exists $u > 0$ such that if $z = x + iy$ with $0 \leq x \leq 1$, $0 < y \leq 1$, then*

$$\mathbb{P}_z^* [E_{z,\delta}] > u.$$

We will use the next lemma in the proof of Lemma 3.5. The lemma may seem to follow immediately from $T_z < \infty$, but it is important to establish uniformity in z .

Lemma 3.6. *For every $\epsilon > 0$, there exists $r < \infty$ such that if $z = x + iy \in \mathbb{H}$,*

$$\mathbb{P}_z^* \{T_z \leq r|z|^2; \quad |U_t| \leq r|z|, 0 \leq t \leq T_z\} \geq 1 - \epsilon. \quad (35)$$

Proof. By scaling and symmetry, it suffices to prove the result for $|z| = 1, x \geq 0$. For fixed z , the result is immediate from the fact that $\mathbb{P}_z^*\{T_z < \infty\} = 1$. It is not difficult to extend this as follows: for every $u > 0$ and every $\epsilon > 0$, there exists r such that if $|z| = 1$ and $S_0(z) \geq u$, then

$$\mathbb{P}_z^*\{T_z \leq r ; |U_t| \leq r, 0 \leq t \leq T_z\} \geq 1 - \epsilon. \quad (36)$$

For $y < 1/100$, let \mathbb{P}_1^* denote probabilities for two-sided chordal SLE_κ to 1. Again, since $\mathbb{P}_1^*\{T_1 < \infty\} = 1$, it is easy to see that for every r , there exists $\epsilon > 0$ such that

$$\mathbb{P}_1^*\{T_1 \leq r ; |U_t| \leq r, 0 \leq t \leq T_1\} \geq 1 - \epsilon. \quad (37)$$

Suppose $z = x + iy \in \mathbb{H}$ with $|z - 1| = \delta/2 < 1/4$. Let

$$\sigma = \inf\{t : |\gamma(t) - 1| = \delta\},$$

and define ξ by $\gamma(\sigma) = 1 + \delta e^{i\xi}$. From (37) and Lemma 2.9, we see that there exists c_1 such that

$$\mathbb{P}_z^*\{\sigma \leq r ; |U_t| \leq r, 0 \leq t \leq \sigma\} \geq 1 - c_1\epsilon.$$

Using Lemma 2.10, we see that there exists $u > 0$, such that

$$\mathbb{P}_z^*\{\sigma \leq r ; |U_t| \leq r, 0 \leq t \leq \sigma ; S_\sigma(z) \geq u\} \geq 1 - 2c_1\epsilon.$$

Using the form of the Poisson kernel in the upper half plane, we can see that $|Z_\sigma(z)| \leq c_2\delta$. By using (36), we see that there exists \tilde{r} such that

$$\begin{aligned} \mathbb{P}_z^*\{\sigma \leq r ; |U_t| \leq r, 0 \leq t \leq \sigma ; S_\sigma(z) \geq u ; T_z - \sigma \leq \tilde{r}\delta^2 ; \\ |U_t - U_\sigma| \leq \tilde{r}\delta, \sigma \leq t \leq T_z\} \geq 1 - 3c_1\epsilon. \end{aligned}$$

□

Proof of Lemma 3.5. By scaling and symmetry, we may assume that $z = x + iy$ with $|z| = 1$ and $x \geq 0, y > 0$. We will show that there exists $c < \infty$ such that for each ρ , there exists $q(\rho) > 0$ such that for all z ,

$$\mathbb{P}_z^*[E_{z, c\sqrt{\rho}}] \geq q(\rho).$$

It suffices to consider $0 < \rho \leq 1/1000$. We will consider two cases: $y \leq 10\rho$ and $y > 10\rho$. In this proof, constants c_1, c_2, \dots are independent of ρ , but constants δ, q_1, q_2, \dots may depend on ρ .

First assume $y \leq 10\rho \leq 1/100$, and hence $3/4 < x \leq 1$. Let

$$\eta = \eta_{\rho, z} = \inf\{t : \operatorname{Re}[\gamma(t)] = x - 4\rho\}.$$

For every $\delta > 0$, consider the event $V_\delta = V_{\delta, \rho, z}$ given by

$$V_\delta = \{\eta \leq \delta ; -\delta \leq U_t \leq 1 + \delta, \quad 0 \leq t \leq \eta\}.$$

Using the deterministic estimate, Lemma 3.4, we can see that by choosing δ sufficiently small, then on the event V_δ , $\text{Im}[\gamma(\eta)] \leq \rho$. By choosing δ smaller if necessary, we assume $\delta < \rho$.

Using Lemma 2.5, we can see that $\mathbb{P}[V_\delta] \geq q_1 > 0$. There is a curve of length at most 11ρ in H_η connecting z and $\gamma(\eta)$. Hence, using the Beurling estimate, there exists c_1 such that

$$|Z_\eta(z)| \leq c_1 \sqrt{\rho}.$$

(Actually, we can get an estimate of $O(\rho)$, but the estimate above suffices for our purposes.) Using Lemma 3.6, we can say there exists c_2 such that

$$\mathbb{P} \left\{ T - \eta \leq c_2 \sqrt{\rho}, \quad \sup_{\eta \leq t \leq T} |U_t - U_\eta| \leq c_2 \sqrt{\rho} \mid V_\delta \right\} \geq \frac{1}{2}.$$

Therefore, with probability at least $q_1/2$,

$$T \leq (c_2 + 1)\sqrt{\rho}, \quad -(1 + c_2)\sqrt{\rho} \leq U_t \leq 1 + (1 + c_2)\sqrt{\rho}.$$

We now assume $y \geq 10\rho$. Let

$$\eta = \eta_{\rho,z} = \inf\{t : \text{Im}[\gamma(t)] = y - 4\rho\}.$$

Let W_t denote a standard Brownian motion and consider the event $E = E_{\delta,x}$ that

$$\begin{aligned} |W_t - (tx/\delta)| &< \delta, \quad 0 \leq t \leq \delta, \\ |W_t - x| &< \delta, \quad \delta \leq t \leq 1/a. \end{aligned}$$

Using standard estimates for Brownian motion (including the Cameron-Martin formula), it is standard to show that for every $\delta > 0$ there exists $u_1 > 0$ such that for all $0 \leq x \leq 1$, $\mathbb{P}(E) \geq u_1$. If we let $U_t = W_t$, then by choosing δ sufficiently small, we see that $\mathbb{P}[E] \geq u_1$.

We claim that there exists $c_3 > 0$ such that on the event E ,

$$S_\eta(z) \geq c_3.$$

To show this, we consider the path $\gamma[0, \eta]$. Let γ^+ be the part of the path mapped to $[U_\eta, \infty)$ under g_η and let γ^- be the part mapped to $(-\infty, U_\eta)$. Using the fact that $\gamma[0, \eta] \subset L_\rho$; $\text{Im}[\gamma(\eta)] = y - 4\rho$ and $\text{Im}[\gamma(t)] < y - 4\rho, t < \eta$, we can see geometrically that there is a positive probability u_2 such that a Brownian motion starting at z exists H_η at γ^+ with probability at least c'_2 and at γ^- with probability at least c'_2 . This combined with (6) gives the lower bound on $S_\eta(z)$. Since $\Upsilon_t(z)$ decreases with t , we get a lower bound on $M_\eta(z)$. Therefore, there exists $q_2 > 0$ such that

$$\mathbb{P}_z^* \{ \gamma[0, \eta] \subset L_{z,\rho} \} \geq q_2.$$

(The reader may note that we have used the trivial bound $\Upsilon_\eta(z) \leq 1$. In fact, $\Upsilon_\eta(z) \asymp \rho$, so we can improve the last estimate but we do not need to. For ρ small, it is much more likely for two-sided radial SLE to follow the L -shape to z than for usual SLE .)

On the event E , there is a curve connecting $\gamma(\eta)$ to z in H_η of length $O(\rho)$. Using the Beurling estimate, we can see that there exists c_4 such that $|Z_\eta(z)| \leq c_4 \sqrt{\rho}$. The proof proceeds as in the previous case.

□

3.2 Remainder of proof

To finish the proof, we need to consider z, w that are close. In this case we will take a stopping time σ such that z, w are not so close in the domain H_σ . The next lemma is easy, but it is useful to state it.

Lemma 3.7. *There exists $c > 0$ such that the following is true. Suppose $z, w \in \mathbb{H}, u \geq 0$ and σ is a stopping time for SLE_κ such that*

$$|\gamma(t) - z| \geq 3|z - w|, \quad 0 \leq t \leq \sigma,$$

and such that

$$\mathbb{P}_z^* \{F_{H_\sigma}(z, w; \gamma(\sigma), \infty) \geq u\} \geq u. \quad (38)$$

Then

$$F(z, w) \geq c u^3.$$

Remark Implicit in the assumptions is $|z| \geq 3|z - w|$. We do not assume that the disk of radius $|z - w|$ about z is contained in \mathbb{H} . In particular, $\text{Im}(z), \text{Im}(w)$ can be small and very different.

Proof. Let $r = |z - w|$ and let \mathcal{B} denote the open disk of radius $2r$ centered at z . By assumption $0 \notin \mathcal{B}$. It is not necessarily the case that $\mathcal{B} \subset \mathbb{H}$; however, for $t \leq \sigma$, the conformal map g_t can be extended to \mathcal{B} by Schwarz reflection.

Let $E = E_u$ denote the event that $F_{H_\sigma}(z, w; \gamma(\sigma), \infty) \geq u$. Using the Beurling estimate, we can see that $M_{t \wedge \sigma}(z), M_{t \wedge \sigma}(w)$ are bounded martingales and hence

$$G(z) = M_0(z) = \mathbb{E}[M_\sigma(z)], \quad G(w) = M_0(w) = \mathbb{E}[M_\sigma(w)].$$

Also, by definition of \mathbb{E}_z^* ,

$$\begin{aligned} \mathbb{E}_z^*[M_\sigma(z) 1_E] &= \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{M_0(z)} = \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{\mathbb{E}[M_\sigma(z)]} \geq u \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{\mathbb{E}[M_\sigma(z) 1_E]} \\ &\geq u \mathbb{E}[M_\sigma(z) 1_E] \geq u^2 G(z). \end{aligned}$$

The first inequality uses (38). Using the distortion theorem on \mathcal{B} , we can see that $|g'_\sigma(z)| \asymp |g'_\sigma(w)|$. We also claim that

$$\frac{S_\sigma(z)}{S_\sigma(w)} \asymp \frac{\text{Im}(z)}{\text{Im}(w)}. \quad (39)$$

To see this, consider the first time that a Brownian motion starting at z, w reaches $\mathbb{R} \cup \partial\mathcal{B}$. If $p(z), p(w)$ denotes the probabilities that the process hits $\partial\mathcal{B} \cap \mathbb{H}$ before leaving \mathbb{H} , then standard estimates (gambler's ruin estimate) show that $p(z)/p(w) \asymp \text{Im}(z)/\text{Im}(w)$. Also, the conditional distributions given that one hits $\partial\mathcal{B}$ are mutually absolutely continuous (here we use either a boundary Harnack principle or the explicit form of the Poisson kernel in a half disk). Given this and (6), we can conclude (39).

Therefore, using (5), we see that

$$\frac{M_\sigma(z)}{M_\sigma(w)} = \frac{G_{H_\sigma}(z; \gamma(\sigma), \infty)}{G_{H_\sigma}(w; \gamma(\sigma), \infty)} \asymp \frac{G(z)}{G(w)},$$

and hence,

$$\mathbb{E}_z^*[M_\sigma(w) 1_E] \geq c u^2 G(w).$$

Also,

$$\mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty) 1_E \mid \mathcal{F}_\sigma] = 1_E u G_{H_\sigma}(w; \gamma(\sigma), \infty) = 1_E u M_\sigma(w).$$

Taking expectations, we get

$$\mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty)] \geq \mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty) 1_E] \geq c u^3 G(w),$$

which implies $F(z, w) \geq c u^3$. \square

Proof of (14). By symmetry and scaling, it suffices to consider $1 = |z| \leq |w|$ with $\operatorname{Re}(z) \geq 0$. If $|w| > 1.01$, then $w \notin L_{z, 1/100}$ and hence Corollary 3.3 implies that $F(z, w) \geq c > 0$.

Similarly, if $\operatorname{Im}(z) > \operatorname{Im}(w) + (1/100)$, then $z \notin L_{w, 1/100}$, and Corollary 3.3 implies that $F(w, z) \geq c$. Using similar facts about real parts and interchanging z, w , it suffices to consider $z = x + iy, w = \tilde{x} + i\tilde{y}$ with $1 \leq |z|, |w| \leq 1.01, x \geq 0$ and

$$y \leq \tilde{y} \leq y + \frac{1}{100}, \quad |x - \tilde{x}| \leq \frac{1}{100}.$$

We now split into two cases: $\tilde{y} \geq 1/10$ and $\tilde{y} < 1/10$.

For $\tilde{y} \geq 1/10$, let $\tau = \inf\{t : \Upsilon_t(z) = 10|z - w|\}, T = T_z$. Using Lemma 2.2, we see that there exists $u > 0$ such that

$$\mathbb{P}_z^*\{S_\tau(z) \geq 1/4\} \geq u.$$

Using distortion estimates and Corollary 3.3, we can see that there exists $c > 0$ such that on the event that $S_\tau(z) \geq 1/4$,

$$F_{H_\tau}(z, w; \gamma(\tau), \infty) \geq c.$$

We can now apply Lemma 3.7.

If $\tilde{y} < 1/10$ and $|z - w| \leq \tilde{y}/20$, we can do similarly as above, interchanging the roles of z and w .

For the remainder, we assume that $\tilde{y} < 1/10$ and $|z - w| \geq \tilde{y}/20$. Note that $x, \tilde{x} > 9/10$. Let

$$r = \max\{\tilde{y}, |z - w|\} < 1/10.$$

Let $\tau = \inf\{t : |\gamma(t) - \tilde{x}| = 4r\}$. If we write

$$\gamma(\tau) = \tilde{x} + 4r e^{i\xi},$$

then by Lemma 2.10 there exists u such that

$$\mathbb{P}_z^*\{\xi > u\} \geq u.$$

On this event, distortion estimates and Corollary 3.3 imply that $F_{H_\tau}(z, w; \gamma(\tau), \infty) \geq c$ for some c (depending on ξ). We can now apply Lemma 3.7. \square

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